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The determinants of some multilevel Vandermonde and Toeplitz matrices

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Abstract
The closed algebraic expressions of the determinants of some multivariate (multilevel) Vandermonde matrices and the associated Toeplitz/Karle–Hauptman matrices are worked out. The formula can usefully be applied to evaluate the determinant of the Karle–Hauptman matrix generated by a principal basic set of reflections, the knowledge of which determines the full diffraction pattern of an ideal crystal.

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1. Introduction

The explicit expression of the Vandermonde determinant has long been known. Recently, there has been interest in evaluating different generalizations of the Vandermonde determinant, including the multivariate (multilevel) case [1, 2], since these matrices and their determinants are useful in many fields, such as crystallography [3–10] or signal theory [2, 11, 12]. The reason why these fields are interested in Vandermonde matrices depends on the fact that the relevant basic problem can be formulated in the one-dimensional (1D) case as follows: to determine the quantities ν_js and x_js (with $0 \leqslant x_j < 1$) defining the distribution $\rho(x) \equiv \sum_{j=1}^{N} \nu_j \delta(x - x_j)$, knowing the latter's Fourier transform $F_h = \sum_{j=1}^{N} \nu_j {\xi_j}^h$ (with $\xi_j \equiv \mathrm{e}^{\mathrm{i}2\pi x_j}$) at an appropriate set of integers h. It is evident that with $h = 0, 1, \ldots, N - 1$ we have a system of N linear equations characterized by a Vandermonde matrix. When the problem is generalized to the more realistic case of a higher dimensional space, i.e. $D \geqslant 2$, the involved matrices become multilevel Vandermonde matrices denoted by $(\mathcal{V})$ in the following.

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Associated with these matrices there are Toeplitz matrices [13, 1], known as Karle–Hauptman matrices in crystallography [3, 4] and having the following structure $(\mathcal{D}) = (\mathcal{V})^{\dagger}(\Delta)(\mathcal{V})$ (see section 3 for precise definition) with (Δ) a diagonal matrix.

The study of these matrices of the kind found in x-ray or neutron scattering from an ideal crystal has helped us to find that their determinants have a simple closed algebraic expression that will be reported here.

2. Generalized multilevel Vandermonde matrix

In this section, we define a generalized D-level Vandermonde matrix in the context of Fourier transforms of weighted discrete sets of points in a D-dimensional space. In the first part of this section we evaluate its determinant in the case $D = 2$; and in the second part, we treat the cases $D > 2$ (section 2.2) that only involve more bookkeeping.

Take a set $\mathcal{S}_D$ of N distinct points in $[0, 1)^D$. In general, some points may have some but not all of their coordinates equal. Then we enumerate the points in $\mathcal{S}_D$ by the different values of the first (*leading*) coordinate, then by the different values of the second (*trailing*) coordinate and so on, employing D indices.

2.1. Case D = 2

For $D = 2$, we have

$$\mathcal{S}_2 \equiv \{\mathbf{r}_{r,s} \equiv (x_r, y_{r,s}) \mid r = 1, \ldots, M, s = 1, \ldots, m_r\}. \tag{1}$$

Here, m_r counts the points which share the same rth value of the first (*leading*) coordinate. Clearly, $1 \leqslant m_r \leqslant N$ whatever r, and $\sum_{r=1}^{M} m_r = N$. Further, in labelling the different x_rs, we choose label r in such a way that

$$m_1 \geqslant m_2 \geqslant \cdots \geqslant m_M. \tag{2}$$

It is observed that set $\mathcal{S}_2$ can bijectively be mapped onto subset $\mathcal{I}_2$

$$\mathcal{I}_2 \equiv \{\mathbf{k} \equiv (h, k) \mid h = 0, \ldots, M-1, k = 0, \ldots, m_{h+1} - 1\} \tag{3}$$

of the $\mathbb{Z}^2$ lattice by putting $h = r - 1$ and $k = s - 1$. We remark that the labelling used in (1) depends on the order in which the coordinates are considered. Therefore there are $D!$ possible different realizations but, in the following, we shall confine ourselves to the one defined above.

Take now a set of N orthonormal vectors $|\mathbf{r}_{r,s}\rangle \equiv |x_r, y_{r,s}\rangle$ ($\mathbf{r}_{r,s} \in \mathcal{S}_2$) forming an orthonormal complete basis of the N-dimensional Hilbert space $\mathcal{H}$. We consider the vectors $|\mathbf{k}\rangle \equiv |(h, k)\rangle$, with $\mathbf{k} \in \mathbb{Z}^2$, defined as

$$|\mathbf{k}\rangle \equiv \sum_{r=1}^{M}\sum_{s=1}^{m_r} \mathrm{e}^{-\mathrm{i}2\pi\mathbf{k}\cdot\mathbf{r}_{r,s}}|\mathbf{r}_{r,s}\rangle \equiv \sum_{r=1}^{M}\sum_{s=1}^{m_r} \mathrm{e}^{-\mathrm{i}2\pi h x_r}\mathrm{e}^{-\mathrm{i}2\pi k y_{r,s}}|x_r, y_{r,s}\rangle. \tag{4}$$

Equivalently, setting

$$\xi_r \equiv \mathrm{e}^{-\mathrm{i}2\pi x_r} \qquad \text{and} \qquad \eta_{r,s} \equiv \mathrm{e}^{-\mathrm{i}2\pi y_{r,s}} \tag{5}$$

we have

$$|\mathbf{k}\rangle = \sum_{r=1}^{M}\sum_{s=1}^{m_r} \xi_r^h \eta_{r,s}^k |x_r, y_{r,s}\rangle. \tag{6}$$

These vectors form a lattice of vectors belonging to $\mathcal{H}$. We restrict now ourselves to the set of the N vectors $|\mathbf{k}\rangle = |(h, k)\rangle$ with $\mathbf{k} \in \mathcal{I}_2$. Then (6) describes a linear transformation between two sets of N vectors of $\mathcal{H}$ that is represented by matrix $(\mathcal{V})$ with elements $\mathcal{V}_{(r,s),(h,k)} \equiv \langle \mathbf{r}_{r,s}|\mathbf{k}\rangle = \xi_r^h \eta_{r,s}^k$. The considered N vectors $|\mathbf{k}\rangle$ are linearly independent iff $(\mathcal{V})$ is non-singular. We write $(\mathcal{V})$ in the factored-block form

$$(\mathcal{V}) = \begin{vmatrix} \xi_1^0(B_{1,1}) & \xi_1^1(B_{1,2}) & \dots & \xi_1^{M-1}(B_{1,M}) \\ \xi_2^0(B_{2,1}) & \xi_2^1(B_{2,2}) & \dots & \xi_2^{M-1}(B_{2,M}) \\ & & \dots & \\ \xi_M^0(B_{M,1}) & \xi_M^1(B_{M,2}) & \dots & \xi_M^{M-1}(B_{M,M}) \end{vmatrix} \tag{7}$$

where the generic $(B_{r,r'})$ element is the $m_r \times m_{r'}$ matrix defined as

$$(B_{r,r'}) \equiv \begin{vmatrix} \eta_{r,1}^0 & \eta_{r,1}^1 & \dots & \eta_{r,1}^{m_{r'}-1} \\ \eta_{r,2}^0 & \eta_{r,2}^1 & \dots & \eta_{r,2}^{m_{r'}-1} \\ & & \dots & \\ \eta_{r,m_r}^0 & \eta_{r,m_r}^1 & \dots & \eta_{r,m_r}^{m_{r'}-1} \end{vmatrix} \tag{8}$$

and $\xi_r^{r'-1}(B_{r,r'})$ denotes the $m_r \times m_{r'}$ matrix obtained by entrywise (element by element) multiplication of matrix $(B_{r,r'})$ by scalar $\xi_r^{r'-1}$. This identifies $(\mathcal{V})$ as a two-level Vandermonde matrix. Clearly, for dimension $D > 2$, we shall have a D-level block structure. Diagonal blocks $(B_{r,r})$ are square Vandermonde matrices and

$$\det(B_{r,r}) = \prod_{1\leqslant s'<s\leqslant m_r} (\eta_{r,s} - \eta_{r,s'}). \tag{9}$$

To facilitate the reader in understanding the way to form matrix $(\mathcal{V})$, we have included a small example in the appendix.

We now show that the analytical expression of the determinant of $(\mathcal{V})$ is also remarkably simple. To this aim, consider the following four points:

(i) The determinant of $(\mathcal{V})$ is a homogeneous polynomial in variables $\{\xi\}$ and $\{\eta\}$. In fact, $\det(\mathcal{V})$ is a sum of terms all having the degrees Q and P in the variables $\{\eta\}$ and $\{\xi\}$, respectively, where

$$Q = \sum_{h=0}^{M-1} \sum_{k=0}^{m_{h+1}-1} k = \frac{1}{2}\left(-N + \sum_{h=0}^{M-1} m_{h+1}^2\right) \tag{10}$$

$$P = \sum_{h=0}^{M-1} h\, m_{h+1}. \tag{11}$$

If, whatever r, $\eta_{r,s} = \eta_{r,s'}$ with $s \neq s'$, two rows of $(\mathcal{V})$ are equal and the determinant must have a zero. This happens for any (s, s') pair and for any r. Hence, using (9), it must result

$$\det(\mathcal{V}) = \mathcal{R}(\{\xi\}) \prod_{r=1}^{M} \det(B_{r,r}). \tag{12}$$

It is noted that the factor $\mathcal{R}$ cannot depend on variables $\{\eta\}$ because the total degree of the expression inside the square brackets is Q. Thus, $\mathcal{R}(\{\xi\})$ is a polynomial of degree P in the $\{\xi\}$ variables.

(ii) Recall now the Laplace expansion of determinants [14, 15]. Evaluate $\det(\mathcal{V})$ by considering the $(m_1 \times m_1)$ minors contained in the first m_1 rows of $(\mathcal{V})$ corresponding to the first row of the matrix reported on the right-hand side of (7). Each of these minors, after factorizing the appropriate powers of ξ common to each column, is a Vandermonde-like matrix having the same exponent sequence in all its rows, the sth of these consisting only powers of $\eta_{1,s}$. The determinants of the minors that do not contain the full set of exponents $0, 1, \ldots, m_1 - 1$ will be 0. Consider now the only minors with nonzero determinant. Recalling the ordering (2), we can say that only minors containing the leftmost $m_2 + 1, \ldots, m_1$th columns of $(\mathcal{V})$ and, thus, of $(B_{1,1})$ will contain exponents $m_2, \ldots, m_1 - 1$ and could therefore be non-singular. Thus, the relevant $(N - m_1) \times (N - m_1)$ complementary minors will never contain the $(m_2 + 1)$ th, $\ldots, m_1$th column of $(\mathcal{V})$.

(iii) Choose now r such that $1 < r \leqslant M$. Assume that $\xi_1 = \xi_r$. We have $m_1 \geqslant m_r$. We can evaluate $\det(\mathcal{V})$ by considering all the minors contained in a rectangular submatrix built by taking the first m_1 rows—all and only those having a power of ξ_1 as factor—and all the m_r rows which have a power of ξ_r as factor. These correspond to the two block rows $B_{1,1}, \ldots, B_{1,M}$ and $B_{r,1}, \ldots, B_{r,M}$ superimposed. Having assumed that $\xi_1 = \xi_r$, each column of one of these minors factorizes ξ_1^h with h in $0, \ldots, M - 1$ depending on the considered column. Thus, we are left with an $(m_1 + m_r) \times (m_1 + m_r)$ Vandermonde matrix whose elements are $\eta_{1,s'}^{k'}$ or $\eta_{r,s}^{k}$ with $1 \leqslant s' \leqslant m_1$ and $1 \leqslant s \leqslant m_r$, while k' and k belong to $0, \ldots, m_1 - 1$. The rank of this matrix can be at most m_1 and, therefore, its determinant will have a zero of order at least equal to $m_r = \min(m_1, m_r)$. Hence, $\det(\mathcal{V})$ evaluated by this procedure will have a zero of order at least equal to m_r owing to the hypothesis $\xi_1 = \xi_r$.

(iv) Assume now that $\xi_2 = \xi_{r'}$ for a particular r' such that $2 < r' \leqslant M$. Before repeating the reasoning made in the case $\xi_1 = \xi_r$, we imagine of having developed $\det(\mathcal{V})$ with respect to the minors contained in the first m_1 rows. As noted above, there is no need to consider complementary minors containing the columns with exponents $m_2, \ldots, m_1 - 1$. Each of these complementary minors can be developed by considering its $(m_2 + m_{r'}) \times (m_2 + m_{r'})$ minors contained in the two blocks rows $B_{2,1}, \ldots, B_{2,M}$ and $B_{r',1}, \ldots, B_{r',M}$, presenting the factors ξ_2 and $\xi_{r'}$, respectively. By the same reasoning made above for the case $\xi_1 = \xi_r$, one concludes that each $(m_2 + m_{r'}) \times (m_2 + m_{r'})$ minor has a rank at most equal to m_2 and that $\det(\mathcal{V})$ has a zero at least of order $m_{r'} = \min(m_2, m_{r'})$ when $\xi_2 = \xi_{r'}$. Thus, the zero of $\det(\mathcal{V})$ is at least of the order $\min(m_r, m_{r'})$ when $\xi_r = \xi_{r'}$. Due to the ordering (2), whenever $r > r'$ we have that $\min(m_r, m_{r'}) = m_r = \mathrm{size}(B_{r,r})$. Thereafter, one can write that

$$\det(\mathcal{V}) = \mathcal{R}_1(\{\eta\}) \prod_{1 \leqslant r' < r \leqslant M} (\xi_r - \xi_{r'})^{\mathrm{size}(B_{r,r})}. \tag{13}$$

The degree in the variables $\{\xi\}$ in the above product is equal to P, so that $\mathcal{R}_1$ is a polynomial of the variables $\{\eta\}$ only. Thus, combining equation (12) with (13), one finds that

$$\mathcal{R}_1(\{\eta\}) = \mathcal{R}_0 \prod_{r=1}^{M} \det(B_{r,r}), \tag{14}$$

where $\mathcal{R}_0$ is a simple constant. Comparing the 'diagonal' term $\prod_{r=1}^{M} \left(\xi_r{}^{(r-1)m_r} \prod_{s=1}^{m_r} \eta_{r,s}^{s-1}\right)$ resulting from the calculation of $\det(\mathcal{V})$, starting from the explicit expression of $(\mathcal{V})$, with the corresponding term obtained developing the products present in (13) combined with (14), one finds that $\mathcal{R}_0 = 1$. Hence, the remarkably simple expression of the determinant of matrix $(\mathcal{V})$ reads

$$\det(\mathcal{V}) = \left(\prod_{1 \leqslant r' < r \leqslant M} (\xi_r - \xi_{r'})^{\mathrm{size}(B_{r,r})} \right) \left(\prod_{r=1}^{M} \det(B_{r,r}) \right). \tag{15}$$

2.2. Case $D > 2$

In the case $D = 3$, the set $\mathcal{S}_3$ of the position vectors will be written as

$$\mathcal{S}_3 \equiv \{ \mathbf{r}_{r,s,t} \equiv (x_r, y_{r,s}, z_{r,s,t}) \mid r = 1, \ldots, M, s = 1, \ldots, M_r, t = 1, \ldots, m_{r,s} \} \tag{16}$$

where M_r counts the different values of the first subleading coordinate of the points sharing the same rth value of the *leading* coordinate while $m_{r,s}$ counts the number of the points of $\mathcal{S}_3$ that have as second coordinate the sth of the possible values when its first coordinate is the rth of the possible values. Clearly, $\sum_{s=1}^{M_r} m_{r,s} = m_r$ where m_r, similarly to definition (1), is the number of the points of $\mathcal{S}_3$ that share the rth leading coordinate so as to have $m_r \geqslant M_r$ and, as before, $N = \sum_{r=1}^{M} m_r$. Labels r and s are now chosen in such a way that $M_1 \geqslant M_2 \geqslant \cdots \geqslant M_M$ and $m_{r,1} \geqslant m_{r,2} \geqslant \cdots \geqslant m_{r,M_r}$. Set $\mathcal{S}_3$ can be bijectively mapped onto subset $\mathcal{I}_3$

$$\mathcal{I}_3 \equiv \{ \mathbf{k} \equiv (h, k, l) \mid h = 0, \ldots, M-1, k = 0, \ldots, M_{h+1} - 1, l = 0, \ldots, m_{h+1,k+1} - 1 \} \tag{17}$$

of the $\mathbb{Z}^3$ lattice. The vectors $|\mathbf{k}\rangle$, with $\mathbf{k} \in \mathbb{Z}^3$, are now defined as

$$|\mathbf{k}\rangle \equiv |(h, k, l)\rangle \equiv \sum_{r=1}^{M} \sum_{s=1}^{M_r} \sum_{t=1}^{m_{r,s}} \mathrm{e}^{-\mathrm{i}2\pi \mathbf{k} \cdot \mathbf{r}_{r,s,t}} |\mathbf{r}_{r,s,t}\rangle \equiv \sum_{r=1}^{M} \sum_{s=1}^{M_r} \sum_{t=1}^{m_{r,s}} \xi_r^h \eta_{r,s}^k \zeta_{r,s,t}^l |x_r, y_{r,s}, z_{r,s,t}\rangle, \tag{18}$$

where

$$\xi_r \equiv \mathrm{e}^{-\mathrm{i}2\pi x_r}; \qquad \eta_{r,s} \equiv \mathrm{e}^{-\mathrm{i}2\pi y_{r,s}}; \qquad \zeta_{r,s,t} \equiv \mathrm{e}^{-\mathrm{i}2\pi z_{r,s,t}}. \tag{19}$$

At the same time, the vectors $|\mathbf{r}_{r,s,t}\rangle \equiv |x_r, y_{r,s}, z_{r,s,t}\rangle$ form an orthonormal complete basis of the N-dimensional Hilbert space $\mathcal{H}$. Take now the N vectors $|\mathbf{k}\rangle = |(h, k, l)\rangle$ with $\mathbf{k} \in \mathcal{I}_3$. The generic element of the 3-level Vandermonde matrix $(\mathcal{V})$, obtained from the sets $\mathcal{S}_3$ and $\mathcal{I}_3$, is $\mathcal{V}_{(r,s,t),(h,k,l)} \equiv \langle \mathbf{r}_{r,s,t} | \mathbf{k} \rangle = \xi_r^h \eta_{r,s}^k \zeta_{r,s,t}^l$. From these considerations, it appears clear how sets $\mathcal{S}_D$ and $\mathcal{I}_D$ are defined if $D > 3$.

It is evident that the expression of the determinant of the corresponding Vandermonde matrix, obtained by $\mathcal{S}_D$ and the associated $\mathcal{I}_D$ for the case $D > 2$, is exactly the same as in (15), with the proviso that now all the $(B_{r,r})$ blocks there present are $(D-1)$-level Vandermonde matrices with (*leading*) variables $\eta_{r,s}$ with $s = 1, \ldots, \mathrm{size}(B_{r,r})$ and sub-blocks $(C_{r;s,p})$ which are again $(D-2)$-level Vandermonde matrices in the trailing variables. To be more explicit, in the case $D = 3$ one has

$$(B_{r,r}) = \begin{vmatrix} \eta_{r,1}^0 (C_{r;1,1}) & \eta_{r,1}^1 (C_{r;1,2}) & \ldots & \eta_{r,1}^{M_r - 1} (C_{r;1,M_r}) \\ \eta_{r,2}^0 (C_{r;2,1}) & \eta_{r,2}^1 (C_{r;2,2}) & \ldots & \eta_{r,2}^{M_r - 1} (C_{r;2,M_r}) \\ & & \ldots & \\ \eta_{r,M_r}^0 (C_{r;M_r,1}) & \eta_{r,M_r}^1 (C_{r;M_r,2}) & \ldots & \eta_{r,M_r}^{M_r - 1} (C_{r;M_r,M_r}) \end{vmatrix} \tag{20}$$

where the symbol $(C_{r;s,p})$ represents the $m_{r,s} \times m_{r,p}$ matrix defined as

$$(C_{r;s,p}) \equiv \begin{vmatrix} \zeta^0_{r,s,1} & \zeta^1_{r,s,1} & \dots & \zeta^{m_{r,p}-1}_{r,s,1} \\ \zeta^0_{r,s,2} & \zeta^1_{r,s,2} & \dots & \zeta^{m_{r,p}-1}_{r,s,2} \\ & & \dots & \\ \zeta^0_{r,s,m_{r,s}} & \zeta^1_{r,s,m_{r,s}} & \dots & \zeta^{m_{r,p}-1}_{r,s,m_{r,s}} \end{vmatrix} \tag{21}$$

and the first index r specifies that sub-blocks $(C_{r;s,p})$ refer to matrix $(B_{r,r})$. This implies that the evaluation of the D-level determinant $\det(\mathcal{V})$ involves a recursive application of (15) down to level 1. In fact, to see that (15) holds in general for any level $D > 2$ we can proceed by induction assuming that (15) holds at level $D-1$ and showing that it still holds at D. For this aim, it is sufficient to trace back steps (i)–(iv).

3. Evaluation of Karle–Hauptman determinants

To write down the explicit form of a Karle–Hauptman matrix $(\mathcal{D})$, it is convenient to re-index sets $\mathcal{S}_D$ and $\mathcal{I}_D$ by unique indices, denoted as ρ and σ, respectively, both ranging over $1, \dots, N$ and preserving the hierarchical order. For $D = 2$, this means

$$\rho \equiv \rho(r,s) \equiv s + \sum_{p=1}^{r-1} m_p; \qquad \sigma \equiv \sigma(h,k) \equiv k + 1 + \sum_{p=0}^{h-1} m_{p+1}. \tag{22}$$

If $D > 2$, the expressions are quite similar. The (σ, σ') element of $(\mathcal{D})$ is defined as

$$\mathcal{D}_{\sigma,\sigma'} \equiv \sum_{\rho=1}^{N} \mathcal{V}^\dagger_{\sigma,\rho} \nu_\rho \mathcal{V}_{\rho,\sigma'} = \sum_{\rho=1}^{N} \nu_\rho \exp(\mathrm{i}2\pi(\mathbf{k}_{\sigma'} - \mathbf{k}_\sigma)\cdot \mathbf{r}_\rho), \qquad \sigma, \sigma' = 1, \dots, N \tag{23}$$

where ν_ρ are given complex numbers. Putting $(\Delta) \equiv \mathrm{diag}\{\nu_1, \nu_2, \dots, \nu_N\}$, we can write

$$(\mathcal{D}) \equiv (\mathcal{V})^\dagger (\Delta)(\mathcal{V}) \tag{24}$$

that clearly coincides with the Karle–Hauptman matrices defined in [3, 4, 8]. At the same time, as the (σ, σ')th element depends only on the difference $\mathbf{k}_{\sigma'} - \mathbf{k}_\sigma$, $(\mathcal{D})$ is a generalized Toeplitz matrix (for $D > 1$, a multilevel Toeplitz matrix). (Considering the case $D = 1$, in fact, (23) is a classical Toeplitz matrix if $\mathbf{k}_{\sigma'} = (\sigma' - 1)$ and $\mathbf{k}_\sigma = (\sigma - 1)$.)

By the results of the previous section and (24) we find that

$$\det(\mathcal{D}) = |\det(\mathcal{V})|^2 \prod_{\rho=1}^{N} \nu_\rho \tag{25}$$

where $\det(\mathcal{V})$ is given by (15). This expression is valid for general complex numbers ν_ρ. Therefore, it applies to the phase problem [3, 4, 10] both for x-ray and for neutrons, and also for anomalous scattering. One concludes that $\det(\mathcal{D})$ is certainly different from zero if no ν_ρ is equal to zero and all $\mathbf{r}_\rho$s are distinct.

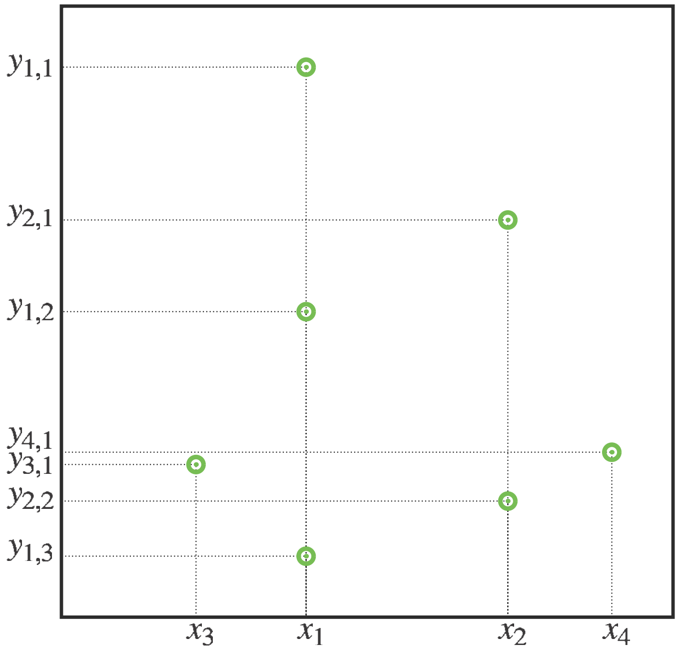

Figure 1. Example of an $\mathcal{S}_2$ set associated with a set of seven distinct points in a 2D space.

4. Conclusion

We have shown that the algebraic expression of the determinant of the multilevel Vandermonde matrix generated by the sets $\mathcal{S}_D$ and $\mathcal{I}_D$ is obtained by the recursive application of (15). Similarly, the determinant of the Toeplitz or Karle–Hauptman matrix defined by equations (23) and (24) is given by (25).

Appendix

Here, we report an example on the construction of a two-level ($D = 2$) Vandermonde matrix. Consider a set $\mathcal{S}_2$ of $N = 7$ points in the $[0, 1) \times [0, 1)$ square, as shown in figure 1. Choosing the horizontal (x) as leading coordinate, we have only $M = 4$ distinct x values. We label the seven points according to their respective x-degeneracy, ordered as

$$\mathcal{S}_2 = \{(x_1, y_{1,1}), (x_1, y_{1,2}), (x_1, y_{1,3}), (x_2, y_{2,1}), (x_2, y_{2,2}), (x_3, y_{3,1}), (x_4, y_{4,1})\}. \tag{A.1}$$

This labelling fulfils the condition in (2), as $m_1 = 3 \geqslant m_2 = 2 \geqslant m_3 = 1 \geqslant m_4 = 1$. Note that *only for the last two points*, having the same degeneracy, we could have exchanged labels. Note also that if we had chosen y as leading coordinate, evidently there would be no degeneracy ($M = N = 7$; $m_r = 1, r = 1, \ldots, 7$) and all labels $(y_r, x_{r,1})$, $r = 1, \ldots, 7$ could have been arbitrarily assigned to the points.

We also assign now the Fourier space ordered point set as in (3),

$$\mathcal{I}_2 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0), (3, 0)\}. \tag{A.2}$$

Matrix $\mathcal{V}$ now is built taking the exponential of $2\pi i$ multiplied by the scalar product of the elements of $\mathcal{S}_2$ (in row order) times the elements of $\mathcal{I}_2$ (in column order). This gives

$$(\mathcal{V}) = \begin{vmatrix} 1 & e^{i2\pi(y_{1,1})} & e^{i2\pi(2y_{1,1})} & e^{i2\pi(x_1)} & e^{i2\pi(x_1+y_{1,1})} & e^{i2\pi(2x_1)} & e^{i2\pi(3x_1)} \\ 1 & e^{i2\pi(y_{1,2})} & e^{i2\pi(2y_{1,2})} & e^{i2\pi(x_1)} & e^{i2\pi(x_1+y_{1,2})} & e^{i2\pi(2x_1)} & e^{i2\pi(3x_1)} \\ 1 & e^{i2\pi(y_{1,3})} & e^{i2\pi(2y_{1,3})} & e^{i2\pi(x_1)} & e^{i2\pi(x_1+y_{1,3})} & e^{i2\pi(2x_1)} & e^{i2\pi(3x_1)} \\ 1 & e^{i2\pi(y_{2,1})} & e^{i2\pi(2y_{2,1})} & e^{i2\pi(x_2)} & e^{i2\pi(x_2+y_{2,1})} & e^{i2\pi(2x_2)} & e^{i2\pi(3x_2)} \\ 1 & e^{i2\pi(y_{2,2})} & e^{i2\pi(2y_{2,2})} & e^{i2\pi(x_2)} & e^{i2\pi(x_2+y_{2,2})} & e^{i2\pi(2x_2)} & e^{i2\pi(3x_2)} \\ 1 & e^{i2\pi(y_{3,1})} & e^{i2\pi(2y_{3,1})} & e^{i2\pi(x_3)} & e^{i2\pi(x_3+y_{3,1})} & e^{i2\pi(2x_3)} & e^{i2\pi(3x_3)} \\ 1 & e^{i2\pi(y_{4,1})} & e^{i2\pi(2y_{4,1})} & e^{i2\pi(x_4)} & e^{i2\pi(x_4+y_{4,1})} & e^{i2\pi(2x_4)} & e^{i2\pi(3x_4)} \end{vmatrix}. \tag{A.3}$$

After passing to variables ξ, η as in (5), with

$$\xi_1 = e^{i2\pi x_1}, \qquad \xi_2 = e^{i2\pi x_2}, \ldots; \qquad \eta_{1,1} = e^{i2\pi y_{1,1}}, \qquad \eta_{1,2} = e^{i2\pi y_{1,2}}, \ldots \tag{A.4}$$

we can write

$$(\mathcal{V}) = \begin{vmatrix} \xi_1^0 \begin{pmatrix} \eta_{1,1}^0 & \eta_{1,1}^1 & \eta_{1,1}^2 \\ \eta_{1,2}^0 & \eta_{1,2}^1 & \eta_{1,2}^2 \\ \eta_{1,3}^0 & \eta_{1,3}^1 & \eta_{1,3}^2 \end{pmatrix} & \xi_1^1 \begin{pmatrix} \eta_{1,1}^0 & \eta_{1,1}^1 \\ \eta_{1,2}^0 & \eta_{1,2}^1 \\ \eta_{1,3}^0 & \eta_{1,3}^1 \end{pmatrix} & \xi_1^2 \begin{pmatrix} \eta_{1,1}^0 \\ \eta_{1,2}^0 \\ \eta_{1,3}^0 \end{pmatrix} & \xi_1^3 \begin{pmatrix} \eta_{1,1}^0 \\ \eta_{1,2}^0 \\ \eta_{1,3}^0 \end{pmatrix} \\ \xi_2^0 \begin{pmatrix} \eta_{2,1}^0 & \eta_{2,1}^1 & \eta_{2,1}^2 \\ \eta_{2,2}^0 & \eta_{2,2}^1 & \eta_{2,2}^2 \end{pmatrix} & \xi_2^1 \begin{pmatrix} \eta_{2,1}^0 & \eta_{2,1}^1 \\ \eta_{2,2}^0 & \eta_{2,2}^1 \end{pmatrix} & \xi_2^2 \begin{pmatrix} \eta_{2,1}^0 \\ \eta_{2,2}^0 \end{pmatrix} & \xi_2^3 \begin{pmatrix} \eta_{2,1}^0 \\ \eta_{2,2}^0 \end{pmatrix} \\ \xi_3^0 \begin{pmatrix} \eta_{3,1}^0 & \eta_{3,1}^1 & \eta_{3,1}^2 \end{pmatrix} & \xi_3^1 \begin{pmatrix} \eta_{3,1}^0 & \eta_{3,1}^1 \end{pmatrix} & \xi_3^2 \begin{pmatrix} \eta_{3,1}^0 \end{pmatrix} & \xi_3^3 \begin{pmatrix} \eta_{3,1}^0 \end{pmatrix} \\ \xi_4^0 \begin{pmatrix} \eta_{4,1}^0 & \eta_{4,1}^1 & \eta_{4,1}^2 \end{pmatrix} & \xi_4^1 \begin{pmatrix} \eta_{4,1}^0 & \eta_{4,1}^1 \end{pmatrix} & \xi_4^2 \begin{pmatrix} \eta_{4,1}^0 \end{pmatrix} & \xi_4^3 \begin{pmatrix} \eta_{4,1}^0 \end{pmatrix} \end{vmatrix} \tag{A.5}$$

where the $(B_{r,r'})$ block matrices are evidenced (cf (7)). By (15) immediately follows that

$$\det(\mathcal{V}) = (\xi_2 - \xi_1)^2(\xi_3 - \xi_1)(\xi_4 - \xi_1)(\xi_3 - \xi_2)(\xi_4 - \xi_2)(\xi_4 - \xi_3) \\ \times (\eta_{1,2} - \eta_{1,1})(\eta_{1,3} - \eta_{1,1})(\eta_{1,3} - \eta_{1,2})(\eta_{2,2} - \eta_{2,1}). \tag{A.6}$$

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